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One-Dimensional Ballistic Aggregation: Rigorous Long-Time Estimates

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Aggregation of mass by perfectly inelastic collisions in a one-dimensional gas of point particles is studied. The dynamics is governed by laws of mass and momentum conservation. The motion between collisions is free. An exact probabilistic description of the state of the aggregating gas is presented. For an initial configuration of equidistant particles on the line with Maxwellian velocity distribution, the following results are obtained in the long-time limit. The probability for finding empty intervals of length growing faster than $t^{2/3}$ vanishes. The mass spectrum can range from the initial mass up to mass of order $t^{2/3}$. Aggregates with masses growing faster than $t^{2/3}$ cannot occur. Our estimates are in accordance with numerical simulations predicting t^{-1} decay for the number density of initial masses and a slower $t^{-2/3}$ decay for the density of aggregates resulting from a large number of collisions (with masses $\sim t^{2/3}$). Our proofs rely on a link between the considered aggregation dynamics and Brownian motion in the presence of absorbing barriers.

KEY WORDS: Inelastic collisions; aggregation dynamics; mass spectrum; long time; Brownian motion.

1. INTRODUCTION

The aim of this paper is to establish some exact results concerning the mechanism of ballistic agglomeration originally studied by Carnevale *et al.*[1] We consider a one-dimensional gas of point particles moving in R^1 and interacting via perfectly inelastic collisions. At the initial moment $t = 0$

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the particles start their motion from points $x_j \in R^1$ with momenta p_j and masses m_j, $j = 0, \pm 1, \pm 2, \ldots$. The numbering is such that

$$x_j < x_{j+1} \tag{1.1}$$

It is also assumed that in any bounded interval there is a finite number of particles. The interaction between the point masses reduces to binary collisions. The motion between collisions is free. In one dimension only the nearest neighbors can collide. When two particles j and $j+1$ undergo the first collision they instantaneously merge, forming a single point mass $m_j + m_{j+1}$ with momentum $p_j + p_{j+1}$. At later times the aggregated masses follow the same dynamical laws. They move freely between the collisions. When two aggregates with masses M_i, M_{i+1} and momenta P_i, P_{i+1} meet they merge and a new point mass $M_i + M_{i+1}$ continues the motion with momentum $P_i + P_{i+1}$. The instantaneous binary collisions are thus perfectly inelastic, governed by the mass and momentum conservation laws. In the course of time the number density of the gas decreases (each collision replaces two particles by a single one). The irreversible rarefication of the system is accompanied by the appearance of massive particles moving slower and slower.

Before engaging in the general analysis let us consider for the sake of introduction a peculiar initial condition allowing for the complete analytic description of the aggregation process. We suppose that at $t = 0$ all particles but one are at rest. The moving particle $j = 0$ starts to propagate from the origin with mass m_0 and momentum $p_0 > 0$. On its way it collides and merges with particles at rest whose mass distribution is described by the density

$$\rho(x) = \sum_{j=1}^{\infty} m_j \delta(x - x_j) \tag{1.2}$$

The mass of the propagating aggregate increases. In order to study the law governing the growth of the mass we consider the equations expressing the momentum conservation

$$P(t) = p_0 \tag{1.3}$$

and conservation of the mass

$$M(t) = m_0 + \int_0^{X(t)} dx\, \rho(x) \tag{1.4}$$

where

$$X(t) = \int_0^t d\tau\, \frac{P(\tau)}{M(\tau)} \tag{1.5}$$

is the position of the aggregate at time t.

Let us examine here a particularly simple case of equally spaced particles with the same mass m,

$$\rho(x) = m \sum_{j=1}^{\infty} \delta(x - ja) \tag{1.6}$$

We also put $m_0 = m$. Equations (1.4) and (1.6) imply the inequality

$$\rho X(t) \leqslant M(t) \leqslant m + \rho X(t) \tag{1.7}$$

where

$$\rho = \frac{m}{a} \tag{1.8}$$

As in the long-time limit $X(t)$ tends to infinity, we asymptotically find

$$M(t) \approx \rho X(t) = \rho p_0 \int_0^t \frac{d\tau}{M(\tau)} \tag{1.9}$$

where the second equality follows from Eq. (1.3). Equation (1.9) can be readily solved yielding the asymptotic law

$$M(t) \approx (2\rho p_0 t)^{1/2} \qquad t \to \infty \tag{1.10}$$

It is interesting to note that the result (1.10) can be simply found by considering the continuum limit

$$m \to 0, \qquad a \to 0, \qquad \rho = \frac{m}{a} = \text{const} \tag{1.11}$$

Applying it to Eq. (1.4), we find that the equality

$$M(t) = \rho X(t) \tag{1.12}$$

holds for any $t > 0$, and thus also the formula $M(t) = (2\rho p_0 t)^{1/2}$. The appearance of the asymptotic law (1.10) for finite times is not surprising, as the formation of mass $M(t) > 0$ in the limit (1.11) requires an infinite number of collisions.

Our object in this paper is to derive some rigorous estimates concerning the long-time behavior of the aggregating gas in the case where all the particles have the same initial momentum distribution. In ref. [1] the growth of the average mass according to the power law $t^{2/3}$ has been found on the basis of postulated universal scaling properties of the ballistic agglomeration. Correspondingly, because of mass conservation, the number density must decrease as $t^{-2/3}$. The $t^{2/3}$ law turns out to be in agreement with numerical studies using molecular dynamics simulations.[1-3] For spatially homogeneous initial distributions the change from $t^{1/2}$ behavior found in (1.10) to the faster $t^{2/3}$ growth of the aggregated mass reflects the passage from essentially one-body dynamics (all particles at rest but one) to a truly many-body interacting system. Mean-field-like theories also have been proposed without, however, any rigorous justification.[2,4] The discussion in ref. 2 refers to the Smoluchowski rate equation for the mass concentrations, whereas the results of ref. 4 rely on a closure of the hierarchy describing the exact dynamics of the system. It thus seems important to provide a rigorous treatment of the aggregation process, and the present paper is a step in this direction.

The general question of the laws governing the dynamics of aggregation of mass has a long history and a big literature. In most cases the motion of particles between inelastic collisions is supposed to follow some stochastic process. To get the flavor of the type of theoretical approaches which have been elaborated we refer the reader to the reviews in refs. 5 and 6. Quite recently some models with no mechanical counterpart have been also studied.[7]

In Section 2, an exact probabilistic description of the state of the aggregating gas at any time t is derived. The formulation is sufficiently general to allow for any choice of the configurational and momentum distribution of the initial particles. We emphasize again that probabilistic aspects enter only through the statistics of initial data. The dynamics is entirely deterministic. Then we specialize the general framework to the model defined by an initial configuration of equidistant masses m [as in (1.6)] and independant Maxwellian velocity distribution. The analysis focuses on probabilities to find in a given interval exactly one particle in a prescribed state, or no particles at all. The main observation is that these probabilities exhibit a remarkable exact scaling property which already singles out the priviledged role of the $t^{2/3}$ power law. Looking for masses of order $t^{2/3}$ at time t is the same as looking for masses of order one with initial masses reduced to $mt^{-2/3}$. Using the bounds established in Section 3, we characterize in Section 4 various aspects of the long-time behavior. More precisely, we estimate the probability to find a mass of order t^{γ}, $\gamma \geqslant 0$, in an interval of fixed length for large time. If $\gamma > 2/3$, we prove that this probability is vanishingly small [in fact $O(\exp(-Ct^{3\gamma-2}), C>0)$]. However, if $2/3 \geqslant \gamma \geqslant 0$, this probability is larger than or equal to $Ct^{-1+\gamma/2}$. The latter conclusion holds also for the mass density. For $\gamma = 2/3$, our result is therfore in agreement with the $t^{2/3}$ law. For $\gamma = 0$, we find that there is still a probability as large as t^{-1} to find aggregates of a fixed number of initial masses. This information is also in accordance with the numerical indications of refs. 2 and 3, where the density of particles with small masses appears to obey a t^{-1} law. In fact, in any interval, we expect to find a whole mass spectum ranging from initial masses m to aggregated masses of order $t^{2/3}$ with increasing probabilities ranging from t^{-1} to $t^{-2/3}$.

As for as the mathematical treatment is concerned, we do not proceed with a direct study of the dynamics in configuration space. The formulation of Section 2 introduces naturally a related stochastic process in momentum space, consisting of the evolution of the momentum of an aggregate in terms of the increase of its mass. When the initial velocity distribution is Maxwellian, it turns out that this process is nothing else than Brownian motion limited by moving barriers. This provides a convenient tool to obtain the estimates of Section 3 (technical parts are relegated to Appendices A and B). In the present situation, these estimates are not sufficient to determine the exact time asymptotics (a more refined control on the excursions of the Brownian motion is needed), and we will come back to this point in future work. The paper ends with concluding remarks on possible generalizations and open questions.

2. PROBABILISTIC FORMULATION

In order to describe quantitatively the effect of the dynamics we shall focus our attention on a finite interval of length $2L$. Let us begin our study by deriving the formula for the probability $\mu^{(0)}(t \mid L)$ that there is no particle within the interval $(-L, L)$ at the time $t>0$. Consider an initial particle j. If at time t is is found, separately or as a part of an aggregate, within the interval $[L, \infty)$, then all the masses to the right of it (particles $j+1$, $j+2$,...) lie also therein. Indeed, the dynamics allows merging, but excludes the possibility of changing the initial linear ordering. Similarly, if particle i is found at time t within the interval $(-\infty, -L]$, the same is also true for all the masses to the left of it (particles $i-1$, $i-2$,...). It follows that the initial states contributing to $\mu^{(0)}(t \mid L)$ are precisely those containing a pair of neighboring particles j and $j+1$ which at time t are found (separately or merged with others) within the intervals $(-\infty, -L]$ and $[L, \infty)$, respectively.

In order to derive a necessary and sufficient condition to find particle $j+1$ within $[L, \infty)$ we consider the motion of the r-particle cluster $\{j+1, j+2,..., j+r\}$. The position of the center of mass of such a cluster will be denoted by X^r_{j+1}. The lower index $j+1$ indicates the particle at the left extremity, and the upper index r gives the number of initial neighboring particles belonging to the cluster. At time t the center of mass occupies the position

$$X^r_{j+1}(t) = \sum_{s=1}^{r} (m_{j+s} x_{j+s} + p_{j+s} t) \Big/ \sum_{s=1}^{r} m_{j+s} \tag{2.1}$$

and lies within the system of point masses which developed from the initial cluster $\{j+1, j+2,..., j+r\}$. Hence, particle $j+1$ starting from the left extremity can never appear to the right of point $X^r_{j+1}(t)$. We conclude that the infinite set of inequalities

$$X^r_{j+1}(t) \geqslant L, \qquad r=1, 2,... \tag{2.2}$$

represents a necessary condition for finding mass $j+1$ within $[L, \infty)$ at time t. It is also a sufficient condition. Indeed, in the event under consideration particle $j+1$ appears at time t isolated or as a part of an aggregate formed by merging with its right neighbors. So its position necessarily coincides with one of the points $X^r_{j+1}(t)$.

In an analogous way one can show that the necessary and sufficient condition to find particle j within the interval $(-\infty, -L]$ is represented by the inequalities

$$X^r_{j-r+1}(t) \leqslant -L, \qquad r=1, 2,... \tag{2.3}$$

where in accordance with our notation $X^r_{j-r+1}(t)$ is the position of the center of mass of the r-particle cluster $\{j-r+1, j-r+2,..., j\}$. Introducing the unit step function

$$\theta(x) = \begin{cases} 1, & x \geqslant 0 \\ 0, & x < 0 \end{cases} \tag{2.4}$$

we can associate with inequalities (2.2) and (2.3) the characteristic function

$$\prod_{r=1}^{\infty} \theta\{X^r_{j+1}(t) - L\}\, \theta\{-L - X^r_{j-r+1}(t)\} \tag{2.5}$$

The infinite product (2.5) equals 1 or 0, depending on whether particles j, $j+1$ are within the intervals $(-\infty, -L]$, $[L, \infty)$, respectively. To different values of j there correspond disjoint sets of initial conditions. The probability $\mu^{(0)}(t \mid L)$ for the absence of particles within the interval $(-L, L)$ at time t is thus given by the formula

$$\mu^{(0)}(t \mid L) = \sum_j \left\langle \prod_{r=1}^{\infty} \theta\{X^r_{j+1}(t) - L\}\, \theta\{-L - X^r_{j-r+1}(t)\} \right\rangle \tag{2.6}$$

where the brackets $\langle \cdots \rangle$ denote the average over the initial distribution of masses, positions, and momenta.

As a next step of our analysis we consider the derivation of the formula for the probability density $\mu_1(X, P, M; t \mid L)$ for finding in the inverval $(-L, L)$ at time t exactly one particle at point X with momentum

P and mass M. Suppose that mass M has been formed by merging of n initial particles $\{j+1, j+2,..., j+n\}$, so that

$$M = m_{j+1} + m_{j+2} + \cdots + m_{j+n} \tag{2.7}$$

Then masses $\{j, j-1,...\}$ and $\{j+n+1, j+n+2,...\}$ must group within the intervals $(-\infty, -L]$ and $[L, \infty)$, respectively. According to the previous analysis, the corresponding characteristic function reads

$$\prod_{r=1}^{\infty} \theta\{-L - X^r_{j-r+1}(t)\}\, \theta\{X^r_{j+n+1}(t) - L\} \tag{2.8}$$

The center of mass of particles $\{j+1, j+2,..., j+n\}$ occupies at time t the position $X^n_{j+1}(t)$ [see Eq. (2.1)] and carries the momentum $p_{j+1} + \cdots + p_{j+n}$. The density μ_1 thus contains the factor

$$\theta(L - |X|)\, \delta(X - X^n_{j+1}(t))\, \delta\left(P - \sum_{s=1}^{n} p_{j+s}\right) \tag{2.9}$$

where δ is the Dirac distribution. We now have to answer the question: what is the necessary and sufficient condition for merging of the n-particle cluster $\{j+1, j+2,..., j+n\}$ into a single point mass before time t? To answer it, let us consider the partition of this cluster into two subsets $\{j+1,..., j+r\}$ and $\{j+r+1,..., j+n\}$, $1 \leqslant r \leqslant n-1$, whose centers of mass are denoted by X^r_{j+1} and X^{n-r}_{j+r+1}, respectively. Suppose now that the trajectories of these centers of mass do not cross before time t, so that

$$X^r_{j+1}(t) < X^{n-r}_{j+r+1}(t) \tag{2.10}$$

Then at time t there are masses with positions not exceeding $X^r_{j+1}(t)$, and there are also masses whose positions are larger than or equal to $X^{n-r}_{j+r+1}(t)$. Hence, if the inequality (2.10) holds, particles $\{j+1,..., j+n\}$ cannot merge in a single mass before time t. If, however, the inverse inequalities

$$X^r_{j+1}(t) \geqslant X^{n-r}_{j+r+1}(t), \qquad r = 1, 2,..., n-1 \tag{2.11}$$

are simultaneously satisfied, a single point mass $M = m_{j+1} + \cdots + m_{j+n}$ must have been formed within the time interval $(0, t]$. Indeed, the occurrence of two or more isolated aggregates would necessarily imply one or more inequalities (2.10). Hence, the set of $n-1$ inequalities (2.11) represents a necessary and sufficient condition for merging of the n particles in a single mass. The characteristic function associated with (2.11) has the form

$$\prod_{r=1}^{n-1} \theta\{X_{j+1}^{r}(t) - X_{j+r+1}^{n-r}(t)\} \tag{2.12}$$

The different values of j and n correspond to different events. So, taking into account the restrictions (2.7), (2.9) and the characteristic functions (2.8), (2.12), we arrive at the formula

$$\begin{aligned} &\mu_1(X, P, M; t \mid L) \\ &\quad = \theta(L - |X|) \left\langle \sum_j \sum_{n=1}^{\infty} \delta\left(M - \sum_{s=1}^{n} m_{j+s}\right) \right. \\ &\qquad \times \delta\left(P - \sum_{s=1}^{n} p_{j+s}\right) \delta(X - X_{j+1}^{n}(t)) \\ &\qquad \times \prod_{r=1}^{\infty} \theta\{-L - X_{j-r+1}^{r}(t)\}\, \theta\{X_{j+n+1}^{r}(t) - L\} \\ &\qquad \left. \times \prod_{s=1}^{n-1} \theta\{X_{j+1}^{s}(t) - X_{j+s+1}^{n-s}(t)\} \right\rangle \end{aligned} \tag{2.13}$$

where again the brackets $\langle \cdots \rangle$ denote the average over the initial ensemble.

In this paper we shall consider a particularly simple initial condition where all the particles have the same mass

$$m_j = m \tag{2.14}$$

and are distributed on a regular lattice with a lattice constant a,

$$x_j = ja \tag{2.15}$$

The initial momenta p_j are supposed to be uncorrelated. The probability density for finding particle j with momentum p_j will be denoted by $\varphi(p_j)$. We assume that φ is a symmetric function

$$\varphi(p) = \varphi(-p) \tag{2.16}$$

The formula (2.1) defining the position of the center of mass of the r-particle cluster $\{j+1, j+2,..., j+r\}$ takes the form

$$X_{j+1}^{r}(t) = (2j + 1 + r)\frac{a}{2} + \frac{1}{rm}\left(\sum_{s=1}^{r} p_{j+s}\right) t \tag{2.17}$$

The characteristic function (2.5) thus takes the form

$$\prod_{r=1}^{\infty} \theta\left\{(2j+1+r)\frac{a}{2}+\frac{1}{rm}\left(\sum_{s=1}^{r} p_{j+s}\right)t-L\right\}$$
$$\times\theta\left\{-L-(2j+1-r)\frac{a}{2}+\frac{1}{rm}\left(\sum_{s=1}^{r} p_{j-r+s}\right)t\right\} \qquad (2.18)$$

In order to evaluate the probability $\mu^{(0)}(t\,|\,L)$ [see Eq. (2.6)] we have to average the function (2.18) over the initial distribution of momenta. The probability for finding particle $j+1$ within $[L, \infty)$ reads

$$\lim_{N\to\infty} \int dp_{j+1}\cdots\int dp_{j+N}\,\varphi(p_{j+1})\cdots\varphi(p_{j+N})$$
$$\times\prod_{r=1}^{N}\theta\left\{\sum_{s=1}^{r} p_{j+s}-\frac{rm}{2\rho t}\left[2\rho(L-(j+1/2)\,a)-rm\right]\right\} \qquad (2.19)$$

where ρ denotes the mass density,

$$\rho=\frac{m}{a} \qquad (2.20)$$

From now on we introduce the total momentum of the r-particle clusters as a new integration variable

$$P_r=\sum_{s=1}^{r} p_{j+s} \qquad (2.21)$$

The probability (2.19) can be then expressed in terms of the function

$$J(Y;t)\equiv\lim_{N\to\infty}\int dP_1\cdots\int dP_N\,\varphi(P_1)\,\varphi(P_2-P_1)\cdots\varphi(P_N-P_{N-1})$$
$$\times\prod_{r=1}^{N}\theta\left\{P_r-\frac{rm}{2\rho t}(2\rho Y-rm)\right\} \qquad (2.22)$$

and it is equal to $J(L-(j+1/2)\,a;t)$. The existence of the limit (2.22) is immediate: it is readily verified that the sequence on the right-hand side of (2.22) is positive and monotonically decreasing as $N\to\infty$. Note that $J(Y;t)\leqslant 1$. In a similar way we can show that in the case of the initial conditions (2.14)–(2.16) the probability for finding particle j within $(-\infty, -L]$ equals $J(L+(j+1/2)\,a;t)$. The general equation (2.6) thus becomes in this case

$$\mu^{(0)}(t\,|\,L)=\sum_j J(L-(j+1/2)\,a;t)\,J(L+(j+1/2)\,a;t) \qquad (2.23)$$

In order to write down the formula for the density $\mu_1(X, P, M; t \mid L)$ we have to consider the average over the initial momenta of the characteristic function (2.12), which takes the form [see Eq. (2.17)]

$$\prod_{r=1}^{n-1}{}' \theta\left\{\sum_{s=1}^{r} p_{j+s} - \frac{r}{n} \sum_{s=1}^{n} p_{j+s} - \frac{rm}{2\rho t}(nm - rm)\right\} \tag{2.24}$$

Introducing again the integration variables (2.21), we find that the average over momenta of the characteristic function (2.24) under the constraint

$$\sum_{s=1}^{n} p_{j+s} = P \tag{2.25}$$

reads

$$I(P, M; t) = \int dP_1 \cdots \int dP_n \, \varphi(P_1)\, \varphi(P_2 - P_1) \cdots \varphi(P_n - P_{n-1})\, \delta(P - P_n)$$
$$\times \prod_{r=1}^{n-1}{}' \theta\left\{P_r - \frac{r}{n} P - \frac{rm}{2\rho t}(M - rm)\right\} \tag{2.26}$$

where $M = nm$. When $n = 1$ Eq. (2.26) reduces to

$$I(P, m; t) = \varphi(P) \tag{2.27}$$

$I(P, M; t)$ is the probability density for the formation of mass M with momentum P out of the initial cluster $\{j+1, j+2, \ldots, j+n\}$.

In terms of the functions J and I the general formula (2.13) for μ_1 in the case of the initial conditions (2.14)–(2.16) becomes

$$\mu_1(X, P, M; t \mid L)$$
$$= \theta(L - |X|) \sum_n \sum_j \left\{\delta(M - nm)\, I(P, nm, t)\right.$$
$$\times \delta\left(X - \frac{P}{nm} t - \left(j + \frac{1}{2}\right) a - \frac{nm}{2\rho}\right)$$
$$\left. \times J\left(L + \left(j + \frac{1}{2}\right) a; t\right) J\left(L - \left(j + \frac{1}{2}\right) a - \frac{nm}{\rho}; t\right)\right\} \tag{2.28}$$

In what follows we shall be interested in the probability to find within the interval $(-L, L)$ exactly one particle whose momentum and mass belong to the intervals $[P_1, P_2]$ and $[M_1, M_2]$, respectively ($P_1 < P_2$,

$M_1 < M_2$). Denoting this probability by $\mu^{(1)}(P_1, P_2; M_1, M_2; t \mid L)$ and using Eq. (2.28) we find[3]

$$\begin{aligned}
&\mu^{(1)}(P_1, P_2; M_1, M_2; t \mid L) \\
&= \int_{-L}^{L} dX \int_{P_1}^{P_2} dP \int_{M_1}^{M_2} dM\, \mu_1(X, P, M; t \mid L) \\
&= \sum_{M_1 \leqslant nm \leqslant M_2} \sum_j \int_{P_1}^{P_2} dP\, \theta\left(L - \left|\left(j+\frac{1}{2}\right) a + \frac{nm}{2\rho} + \frac{Pt}{nm}\right|\right) \\
&\quad \times I(P, nm; t)\, J\left(L + \left(j+\frac{1}{2}\right) a; t\right) J\left(L - \left(j+\frac{1}{2}\right) a - \frac{nm}{\rho}; t\right) \qquad (2.29)
\end{aligned}$$

The analysis presented above can be generalized in a straighforward way to derive the formula for the probability density μ_k for finding in the interval $(-L, L)$ at time t exactly k particles with masses M_i, momenta P_i, and positions X_i, $i = 1, ..., k$. Adopting the initial conditions (2.14)–(2.16), one finds for $X_1 < X_2 < \cdots < X_k$

$$\begin{aligned}
&\mu_k(X_1, P_1, M_1, ..., X_k, P_k, M_k; t \mid L) \\
&= \theta(X_1 + L) \prod_{i=1}^{k-1} \theta(X_{i+1} - X_i)\, \theta(L - X_k) \\
&\quad \times \sum_j \sum_{n_1} \cdots \sum_{n_k} \prod_{i=1}^{k} \Bigg\{ \delta(M_i - n_i m)\, I(P_i, n_i m; t) \\
&\quad \times \delta\left(X_i - \frac{P_i}{n_i m} t - \left(j+\frac{1}{2}\right) a - \left(n_1 + \cdots + n_{i-1} + \frac{n_i}{2}\right) \frac{m}{\rho}\right)\Bigg\} \\
&\quad \times J\left(L + \left(j+\frac{1}{2}\right) a; t\right) J\left(L - \left(j+\frac{1}{2}\right) a - (n_1 + \cdots + n_k) \frac{m}{\rho}; t\right) \qquad (2.30)
\end{aligned}$$

Equation (2.30) shows that all distributions μ_k, $k = 1, 2, ...$, can be expressed in terms of the two functions $I(P, M; t)$ and $J(Y; t)$. The next section will be devoted to the study of their properties. One could use Eq. (2.30) to calculate the number density $F(X_1, P_1, M_1; t)$ of particles with given mass and momentum as well as higher-order correlation functions. The exact formula involves, however, all distributions μ_k,[8]

$$F(1; t) = \mu_1(1; t \mid L) + \sum_{k=2}^{\infty} k \int d2 \cdots \int dk\, \mu_k(1, 2, ..., k \mid L) \qquad (2.31)$$

[3] $\mu^{(k)}$ denotes the probabilities, while μ_k refers to probability densities.

where $j \equiv (X_j, P_j, M_j)$, $j = 1, 2, \ldots$. We shall not analyze this complicated problem here, focusing our attention on the simplest probabilities $\mu^{(0)}(t \mid L)$ and $\mu_1(X, P, M; t \mid L)$.

3. RANDOM WALKS IN MOMENTUM SPACE

As already remarked, the description of the state of the aggregating gas constituted initially of equal masses distributed on a regular lattice requires the knowledge of just two functions, $I(P, M; t)$ and $J(Y; t)$, defined by Eqs. (2.26) and (2.22), respectively. It turns out that their properties can be conveniently analyzed with the concepts of one-dimensional random walks with absorbing barriers.

Let $P_n = p_1 + \cdots + p_n$ be the random walk in momentum space starting at $P_0 = 0$, defined as the sum of independent and identically distributed momenta p_r, $r = 1, 2, \ldots, n$, with distribution $\varphi(p)$; P_n represents the total momentum of a mass $M = nm$ which has been constituted by merging of n initial masses m. The epochs of this random walk are not labeled by the physical time t, but rather by the sequence of increasing masses $\tau_r = mr$, $r = 1, 2, \ldots, n$, of initial r-particle clusters moving with momentum $P(\tau_r) = P_r = p_1 + \cdots + p_r$. With this concept in mind the structure of Eq. (2.26) permits us to interpret $I(P, M; t)\, dP$ as the probability for a random walk starting from $P_0 = 0$ at $\tau_0 = 0$ to end between P and $P + dP$ at the epoch $\tau_n = nm = M$, overcoming the barrier

$$P(\tau_r) \geqslant \tau_r \left[\frac{P}{M} + \frac{M - \tau_r}{2\rho t} \right], \qquad r = 1, 2, \ldots, n - 1 \tag{3.1}$$

Similarly, the function $J(Y; t)$ can be looked upon as the probability for a random walk starting from $P_0 = 0$ at the epoch $\tau_0 = 0$ to stay above the barrier

$$P(\tau_r) \geqslant \tau_r \frac{2\rho Y - \tau_r}{2\rho t}, \qquad r = 1, 2, \ldots \tag{3.2}$$

for all subsequent epochs.

The study of such discrete-"time" random walks for general distributions $\varphi(p)$ is a nontrivial problem. To make progress we assume from now on that the initial distribution of momentum corresponds to thermal equilibrium at inverse temperature β,

$$\varphi_m(p) = (\beta/2\pi m)^{1/2} \exp(-\beta p^2/2m) \tag{3.3}$$

In this case the change of variables

$$P_r \to P_r + \frac{r}{n} P \tag{3.4}$$

in Eq. (2.26) yields the relation

$$I(P, M; t) = \exp(-\beta P^2/2M)\, I(P=0, M; t) \tag{3.5}$$

It will thus be sufficient to study the function

$$I_m(0, M; t)$$
$$= \int dP_1 \cdots \int dP_{n-1}\, \varphi_m(P_1)\, \varphi_m(P_2 - P_1) \cdots \varphi_m(P_{n-1} - P_{n-2})\, \varphi_m(P_{n-1})$$
$$\times \prod_{r=1}^{n-1} \theta \left\{ P_r - \frac{rm}{2\rho t} (M - rm) \right\} \tag{3.6}$$

We have added the index m to $I(0, M; t)$ to make explicit its dependence on the size of the initial mass. In the whole subsequent analysis the mass density $\rho = m/a$ is considered as a fixed parameter. Moreover, in Eqs. (2.22) and (2.26) ρ enters exclusively through the combinations ρt and ρY. A change of ρ is equivalent to a change of scale for the variables t and Y, and thus we can put $\rho = 1/2$ without loss of generality. Similarly, we shall use the notation $J_m(Y; t)$ for the function (2.22).

The choice of the Maxwell distribution (3.3) implies the following basic scaling properties of the functions I_m and J_m:

$$I_m(0, M; t) = t^{-1/3} I_{mt^{-2/3}}(0, Mt^{-2/3}; 1) \tag{3.7}$$

$$J_m(Y; t) = J_{mt^{-2/3}}(Yt^{-2/3}; 1) \tag{3.8}$$

The equalities (3.7) and (3.8) follow directly from the identity

$$\theta\left(P_r - \frac{rm}{t} (M - rm) \right) = \theta(P_r t^{-1/3} - rmt^{-2/3}(Mt^{-2/3} - rmt^{-2/3})) \tag{3.9}$$

and the relation

$$\varphi_m(P) = t^{-1/3} \varphi_{mt^{-2/3}}(Pt^{-1/3}) \tag{3.10}$$

Equations (3.7) and (3.8) show the priviledged role of masses and distances of order $t^{2/3}$ in the dynamics of aggregation. In particular, Eq. (3.7) implies that looking for the formation of masses M of order $t^{2/3}$ at time t is equivalent to looking for the presence of masses of order one at time $t = 1$ with the initial masses scaled down to $mt^{-2/3}$. As the mass density $\rho = m/a$ is kept constant, this implies also the scaling of the interparticle distance to $at^{-2/3}$, and thus the approach to the continuum limit. Owing to the scaling properties (3.7) and (3.8), we can further reduce our analysis to the study of two functions

$$\begin{aligned} I_m(M) &= I_m(P=0, M; t=1) \\ J_m(Y) &= J_m(Y; t=1) \end{aligned} \tag{3.11}$$

The main observation at this point is that the functions $I_m(M)$ and $J_m(Y)$ can be expressed in terms of conditional Wiener measures of appropriate sets of Brownian paths $P(\tau)$. Indeed $\varphi_m(P_{r+1} - P_r)$ is nothing else than the distribution of the increment of the Brownian path $P(\tau_{r+1}) - P(\tau_r)$ corresponding to the "time" step $\tau_{r+1} - \tau_r = m$. The set corresponding to $I_m(M)$ is composed of paths starting and ending at the origin, $P(0) = P(M) = 0$, and being above the parabolic barrier

$$f_M(\tau) = \tau(M - \tau) \tag{3.12}$$

at discrete "times" $\tau_r = rm$, $r = 1, 2,..., n-1$, $M = nm$. We thus find

$$I_m(M) = E_w(\{P(\tau_r) \geq f_M(\tau_r), r = 1,..., n-1\} \mid P(0) = 0; P(M) = 0) \tag{3.13}$$

where $E_w(\{\cdots\} \mid P(M_1) = P_1;\ P(M_2) = P_2)$ denotes the expectation value with respect to the conditional Wiener measure for paths starting from P_1 at "time" M_1 and ending in P_2 at "time" M_2. In the same way we identify $J_m(Y)$ as the measure of paths starting from $P(0) = 0$ and being above the barrier $f_Y(\tau) = \tau(Y - \tau)$ for all discrete "times" $\tau_r = rm$, $r = 1, 2,...$. The corresponding formula reads

$$J_m(Y) = \lim_{N \to \infty} \int dP\, E_w(\{P(\tau_r) \geq f_Y(\tau_r), r = 1,..., N\} \mid P(0) = 0; P(\tau_N) = P) \tag{3.14}$$

It is clear from the scaling relations (3.7), (3.8) that the long-time behavior of the aggregation process is governed by the asymptotic properties of the functions I_m, J_m for m going to zero. So in the next section we derive a number of bounds on I_m, J_m which will be the basis for the discussion of the time asymptotics of probabilities $\mu^{(0)}$ and $\mu^{(1)}$.

4. LOWER AND UPPER BOUNDS

Equations (3.13) and (3.14) permit us to derive bounds on the relevant functions $I_m(M)$ and $J_m(Y)$. Upper bounds are readily found by dropping some of the constraints in Eqs. (3.13) and (3.14). Assuming for convenience

n even and keeping only the requirement for the paths to overcome the highest point $f_M(M/2) = M^2/4$ of the barrier (3.12), we find the inequality

$$I_m(M) \leqslant E_w(\{P(M/2) \geqslant f_M(M/2)\} \mid P(0)=0;\, P(M)=0)$$
$$= (\beta/\pi M) \int_{M^2/4}^{\infty} dP \exp(-2\beta P^2/M) \leqslant (C_1/\sqrt{M}) \exp(-C_2 M^3) \quad (4.1)$$

where C_1, C_2 are positive constants (independent of m and M). In the same way when $Y > 0$ we derive the upper bound

$$J_m(Y) \leqslant \lim_{N \to \infty} \int dP\, E_w(\{P(Y/2) \geqslant Y^2/4\} \mid P(0)=0;\, P(\tau_N)=P)$$
$$= (\beta/\pi Y)^{1/2} \int_{Y^2/4}^{\infty} dP \exp(-\beta P^2/Y) \leqslant C_1 \exp(-C_2 Y^3) \quad (4.2)$$

uniformly with respect to m.

In order to obtain lower bounds, we strengthen the constraints in Eqs. (3.13) by requiring that the paths remain above the barrier $f_M(\tau)$ for all τ in the interval $m \leqslant \tau \leqslant M-m$, and not only in the discrete points τ_r. Thus

$$I_m(M) \geqslant E_w(\{P(\tau) \geqslant f_M(\tau),\, m \leqslant \tau \leqslant M-m\} \mid P(0)=0;\, P(M)=0)$$
$$= \int_{f_M(m)}^{\infty} dP_1 \int_{f_M(M-m)}^{\infty} dP_{n-1}\, \varphi_m(P_1)$$
$$\times K(m, P_1;\, M-m, P_{n-1})\, \varphi_m(P_{n-1}) \quad (4.3)$$

where

$$K(M_1, P_1;\, M_2, P_2)$$
$$= E_w(\{P(\tau) \geqslant f_M(\tau),\, M_1 \leqslant \tau \leqslant M_2\} \mid P(M_1)=P_1,\, P(M_2)=P_2) \quad (4.4)$$

is the measure of the set of paths starting from P_1 at "time" M_1, ending in P_2 at M_2, and remaining above $f_M(\tau)$ for all "times" $M_1 \leqslant \tau \leqslant M_2$. Performing in Eq. (2.17) the change of variables $P_1 = \sqrt{m}\, u$, $P_{n-1} = \sqrt{m}\, v$, we eventually find

$$I_m(M) \geqslant \int_{\sqrt{m}(M-m)}^{\infty} du\, \varphi(u) \int_{\sqrt{m}(M-m)}^{\infty} dv\, \varphi(v)\, K(m, \sqrt{m}\, u;\, M-m, \sqrt{m}\, v) \quad (4.5)$$

where

$$\varphi(u) = (\beta/2\pi)^{1/2} \exp(-\beta u^2/2) \quad (4.6)$$

It is shown in Appendix A that as a consequence of the Feynman–Kac formula the "propagator" $K(M_1, P_1; M_2, P_2)$ is proportional to the integral kernel of the semigroup generated by the differential operator

$$H_D = -\frac{1}{2\beta}\frac{d^2}{dZ^2} + 2\beta Z \tag{4.7}$$

on $L^2([0, \infty), dZ)$ with Dirichlet boundary conditions at $Z=0$. This implies that

$$K(0, P_1 = 0; M, P_2) = K(0, P_1; M, P_2 = 0) = 0 \tag{4.8}$$

and thus for $M > 0$ and m small enough

$$K(m, \sqrt{m}\, u; M-m, \sqrt{m}\, v) = muvG(M) + O(m^{3/2}) \tag{4.9}$$

where

$$G(M) = \frac{\partial^2}{\partial P_1\, \partial P_2} K(0, P_1; M, P_2)\,|_{P_1 = P_2 = 0} \tag{4.10}$$

is strictly positive for any $M > 0$ [see Appendix B for the justification of (4.9) and an explicit lower bound for $G(M)$]. Expanding the right-hand side of (4.5) with the help of (4.9) and keeping the lowest order terms, we find

$$I_m(M) \geqslant \frac{\beta m}{2\pi} G(M) + O(m^{3/2}) \tag{4.11}$$

The estimate of the remainder in (4.11) is uniform with respect to M for $M \geqslant M_0 > 0$.

In the same way we find the lower bound for $J_m(Y)$ by requiring that the paths stay above $f_Y(\tau)$ for all $\tau \geqslant m$

$$\begin{aligned} J_m(Y) &\geqslant \lim_{N\to\infty} \int dP\, E_w(\{P(\tau) \geqslant f_Y(\tau), m \leqslant \tau \leqslant Nm\} \mid P(0) = 0; P(Nm) = P) \\ &= \int_{f_Y(m)}^{\infty} dP_1\, \varphi_m(P_1) \lim_{N\to\infty} \int_{f_Y(Nm)}^{\infty} dP\, K(m, P_1; Nm, P) \\ &= \int_{\sqrt{m}(Y-m)}^{\infty} du\, \varphi(u) \lim_{N\to\infty} \int_{Nm(Y-Nm)}^{\infty} dP\, K(m, \sqrt{m}\, u; Nm, P) \end{aligned} \tag{4.12}$$

Using (4.8) again, we see that $K(m, \sqrt{m}\, u; Nm, P)$ with $Nm = M$ fixed is of the order $\sqrt{m}$ as $m \to 0$, and the same will thus be true for the lower bound for $J_m(Y)$. More precisely, it is shown in Appendix B that

$$J_m(Y) \geqslant (\beta m/2\pi)^{1/2} S(Y) + mR(m, Y) \tag{4.13}$$

When $Y \leqslant 0$ one has $S(Y) = 2\beta\,|Y|$ and $R(m, Y) = O((|Y| + 1)^2)$, while for $Y > 0$ both $S(Y)$ and $R(m, Y)$ are $O(\exp(-CY^3))$. The estimates of $R(m, Y)$ are uniform with respect to m for m small enough. From (4.13) and the above remarks one can derive the lower bound for the product $J_m(Y_1 - Y)\ J_m(Y_2 + Y)$ appearing in Eqs. (2.23), (2.29), defining the probabilities $\mu^{(0)}$ and $\mu^{(1)}$. One finds

$$J_m(Y_1 - Y)\, J_m(Y_2 + Y) \geqslant \frac{\beta m}{2\pi} S(Y_1 - Y)\, S(Y_2 + Y) + m^{3/2} O(\exp(-C\,|Y|^3)) \tag{4.14}$$

provided Y_1 and Y_2 are bound to compact sets.

5. LONG-TIME CHARACTERISTICS

The estimates of functions I_m and J_m now will be used to derive asymptotic properties of probabilities $\mu^{(0)}$ and $\mu^{(1)}$ in the long-time limit. Let us begin the discussion by considering the probability $\mu^{(0)}(t\,|\,L)$, given by Eq. (2.23), to observe an empty interval $(-L, L)$ at time t. The scaling property (3.8) suggests that intervals of length $2L \approx t^{2/3}$ play a special role in the dynamics of the gas. We shall show now that this is indeed the case. To this end let us first study the behavior of the probability $\mu^{(0)}$ in the case where $L = \bar{L} t^{2/3}$, for fixed $\bar{L}$. Using the scaling (3.8), we rewrite Eq. (2.23) in the form

$$\mu^{(0)}(t\,|\,L) = \sum_j J_{\bar{m}}(\bar{L} - (j + 1/2)\,\bar{a})\, J_{\bar{m}}(\bar{L} + (j + 1/2)\,\bar{a}) \tag{5.1}$$

where

$$\bar{m} = m t^{-2/3}, \qquad \bar{a} = a t^{-2/3} \tag{5.2}$$

The inequality (4.14) yields the estimate

$$\mu^{(0)}(t\,|\,\bar{L} t^{2/3}) \geqslant (\beta \bar{m}/2\pi) \sum_j S(\bar{L} - (j + 1/2)\,\bar{a})\, S(\bar{L} + (j + 1/2)\,\bar{a}) + \bar{m}^{1/2} O\left(\bar{m} \sum_j \exp[-C\,|(j + 1/2)\,\bar{a}|^3]\right) \tag{5.3}$$

As already noted, looking for L of the order $t^{2/3}$ amounts to scaling down the initial mass m and spacing a to the infinitesimal quantities $\bar{m}$ and $\bar{a}$. Since the bound (4.14) provides the factor $\bar{m} = \rho \bar{a} = \bar{a}/2$ in front of the sums, it is clear that for $t \to \infty$ the right-hand side of inequality (5.3) converges to

$$\frac{\beta}{4\pi} \int_{-\infty}^{+\infty} dY\, S(\bar{L} - Y)\, S(\bar{L} + Y) > 0 \tag{5.4}$$

We conclude that in the long-time limit

$$\mu^{(0)}(t \mid \bar{L} t^{2/3}) \geqslant C > 0 \tag{5.5}$$

This means that there is a nonvanishing probability to find empty intervals of the order $t^{2/3}$.

The particular role of the length scale $t^{2/3}$ appears in the fact that for larger intervals

$$L = \bar{L} t^{\alpha}, \qquad \alpha > 2/3 \tag{5.6}$$

the probability $\mu^{(0)}$ asymptotically rapidly vanishes. To show this we use $J_m \leqslant 1$, the scaling (3.8), and the upper bound (4.2), finding

$$\begin{aligned} \mu^{(0)}(t \mid L) &\leqslant 2 \sum_{j=0}^{\infty} J_m(L - (j + 1/2)\, a; t)\, J_m(L + (j + 1/2)\, a; t) \\ &\leqslant 2 \sum_{j=0}^{\infty} J_{\bar{m}}(L t^{-2/3} + (j + 1/2)\, \bar{a}) \\ &\leqslant 2C_1 \sum_{j=0}^{\infty} \exp\{-C_2[L t^{-2/3} + (j + 1/2)\, \bar{a}]^3\} \\ &\leqslant 2C_1 \exp\{-C_2 (L t^{-2/3})^3\} \sum_{j=0}^{\infty} \exp\{-C_2[(j + 1/2)\, \bar{a}]^3\} \end{aligned} \tag{5.7}$$

The sum in the last term behaves like $\bar{a}^{-1} \sim t^{2/3}$ for t large. We thus see that (5.7) tends to zero as $t \to \infty$ when the condition (5.6) if fulfilled. More precisely, we conclude

$$\mu^{(0)}(t \mid \bar{L} t^{\alpha}) = O(\exp\{-C t^{3\alpha - 2}\}) \qquad \text{for} \quad \alpha > 2/3 \tag{5.8}$$

The estimate (5.8) implies that the intervals of the order t^{α}, $\alpha > 2/3$, certainly contain particles. In view of the previous result (5.5), $\alpha = 2/3$ appears as a threshold value. We can conclude that although a typical (average) distance between the particles can be of the order $t^{2/3}$, it cannot be larger than that.

Let us turn now to the study of the probability $\mu^{(1)}$ given by Eq. (2.29). An interesting question is what size of masses one can expect to observe in the long-time regime $t \to \infty$. To begin with, we analyze the possibility of finding a mass of the order t^{β} with $\beta > 2/3$. Using Eqs. (3.5) and (3.7), and the fact that J_m can be majorized by 1, and extending the momentum integration to the whole range $(-\infty, +\infty)$, we find from Eq. (2.29) the inequality

$$\mu^{(1)}(P_1, P_2; M_1, M_2; t \mid L) \leqslant \frac{2L}{a} (\beta/2\pi)^{1/2} \sum_{M_1 \leqslant nm \leqslant M_2} (nmt^{-2/3})^{1/2} I_{mt^{-2/3}}(nmt^{-2/3}) \tag{5.9}$$

The straightforward application of the upper bound (4.1) yields the estimate

$$\mu^{(1)}(P_1, P_2; M_1, M_2; t \mid L) \leqslant C_1 \frac{2L}{a} \left(\frac{M_2 - M_1}{m} \right) \exp\{ -C_2 (M_1 t^{-2/3})^3 \} \tag{5.10}$$

Hence, when $M_1, M_2 \sim t^{\beta}$, $\beta > 2/3$, the probability tends to zero in the limit $t \to \infty$ for arbitrarily large intervals $2L \sim t^{\alpha}$, $\alpha \geqslant 0$, owing to the exponential factor $\exp(-Ct^{3\beta - 2})$. The important conclusion is that the process of aggregation has asymptotically zero probability weight to produce in any interval masses larger than $t^{2/3}$.

We consider now the probability of finding a particle with mass of the order $t^{2/3}$ and momentum of the order $t^{1/3}$. Using the scaling (3.7), (3.8), we rewrite Eq. (2.29) in the form

$$\begin{aligned} &\mu^{(1)}(\bar{P}_1 t^{1/3}, \bar{P}_2 t^{1/3}; \bar{M}_1 t^{2/3}, \bar{M}_2 t^{2/3}; t \mid L) \\ &\quad = \sum_{\bar{M}_1 \leqslant n\bar{m} \leqslant \bar{M}_2} \sum_j \int_{\bar{P}_1}^{\bar{P}_2} dP \exp\{ -\beta P^2 / 2n\bar{m} \} \, I_{\bar{m}}(n\bar{m}) \\ &\quad \times \theta(Lt^{-2/3} - |(j + 1/2)\, \bar{a} + n\bar{m} + P/n\bar{m}|) \\ &\quad \times J_{\bar{m}}(Lt^{-2/3} + (j + 1/2)\, \bar{a}) \, J_{\bar{m}}(Lt^{-2/3} - (j + 1/2)\, \bar{a} - 2n\bar{m}) \end{aligned} \tag{5.11}$$

where $\bar{P}_1$, $\bar{P}_2$, and $\bar{M}_2 > \bar{M}_1 > 0$ are fixed and $\bar{m} = mt^{-2/3}$. Application of the lower bounds (4.11) and (4.14) introduces the product of infinitesimals $\bar{m}^2 = \bar{m}\bar{a}/2$ in front of the double summation. So, when $t \to \infty$ the right-hand side of Eq. (5.11) is larger than or equal to

$$\begin{aligned} &\frac{1}{2} \left(\frac{\beta}{2\pi} \right)^2 \int dX \int_{\bar{M}_1}^{\bar{M}_2} dM \int_{\bar{P}_1}^{\bar{P}_2} dP \, \theta\left(Lt^{-2/3} - \left| X + M + \frac{P}{M} \right| \right) \exp\left(\frac{-\beta P^2}{2M} \right) \\ &\quad \times G(M) \, S(Lt^{-2/3} + X) \, S(Lt^{-2/3} - X - 2M) \end{aligned} \tag{5.12}$$

Introducing a new integration variable $Y = X + M + P/M$, we then arrive at the long-time estimate

$$\mu^{(1)}(\bar{P}_1 t^{1/3}, \bar{P}_2 t^{1/3}; \bar{M}_1 t^{2/3}, \bar{M}_2 t^{2/3}; t \mid L)$$
$$\geqslant \frac{1}{2}\left(\frac{\beta}{2\pi}\right)^2 \int_{-\bar{L}}^{\bar{L}} dY \int_{\bar{M}_1}^{\bar{M}_2} dM \int_{\bar{P}_1}^{\bar{P}_2} dP \exp\left(\frac{-\beta P^2}{2M}\right) G(M)$$
$$\times S\left(\bar{L} - M - \frac{P}{M} + Y\right) S\left(\bar{L} - M + \frac{P}{M} - Y\right) \tag{5.13}$$

where $\bar{L} = Lt^{-2/3}$. The main conclusion from (5.13) is that there is a nonvanishing probability to find particles with momentum $\sim t^{1/3}$ and mass $\sim t^{2/3}$ in the intervals of length $\sim t^{2/3}$. If we observe particles in a fixed interval of length $2L$, the appropriate form of (5.13) is

$$\mu^{(1)}(\bar{P}_1 t^{1/3}, \bar{P}_2 t^{1/3}; \bar{M}_1 t^{2/3}, \bar{M}_2 t^{2/3}; t \mid L) \geqslant C \frac{2L}{t^{2/3}} \tag{5.14}$$

The estimate (5.13)–(5.14) dealt with the probability of finding large masses $M \sim t^{2/3}$ as $t \to \infty$. We may also ask about the asymptotic survival of an initial mass m. To discuss this point, we first provide a lower bound for the probability $\mu^{(1)}(m; t \mid L)$ of finding a mass m in $(-L, L)$ that has not collided up to time t. For this, we start from the formula (2.29) where we keep only the term $n = 1$ and integrate over all momenta. Using the scaling property (3.8) and the lower bound (4.14) and replacing asymptotically the j summation by an integral gives

$$\mu^{(1)}(m; t \mid L)$$
$$\geqslant \int dX \int dp\, \theta\left(Lt^{-2/3} - \left|X + \frac{p}{m} t^{1/3}\right|\right)$$
$$\times \left(\frac{\beta}{2\pi}\right)^{3/2} \exp\left(\frac{-\beta p^2}{2m}\right) [S(Lt^{-2/3} + X)\, S(Lt^{-2/3} - X) + O(t^{-1/3})] \tag{5.15}$$

Performing the change of variables $p = mut^{-1/3}$, $X = Y + u$, one finds that the right-hand side of (5.15) behaves as (up to a constant factor)

$$\frac{2L}{t} \int_{-\infty}^{\infty} du\, S(u)\, S(-u), \qquad t \to \infty \tag{5.16}$$

Therefore we conclude that there exists a positive constant C such that

$$\mu^{(1)}(m; t \mid L) \geqslant C \frac{2L}{t} \tag{5.17}$$

for t sufficiently large.

More generally, we can estimate the probability $\mu^{(1)}(\bar{M}_1 t^\gamma, \bar{M}_2 t^\gamma; t \mid L)$ of finding a mass of order t^γ, $0<\gamma<2/3$, in $(-L, L)$. Its lower bound is the same as (5.13), replacing there the mass integration limits by $\bar{M}_j t^{\gamma-2/3}$, $j=1$, 2, $\bar{M}_2>\bar{M}_1>0$, and extending the momentum integration to $(-\infty, +\infty)$. After the change of variable $P=Mu$, this bound becomes

$$\frac{1}{2}\left(\frac{\beta}{2\pi}\right)^2 \int_{-Lt^{-2/3}}^{Lt^{-2/3}} dY \int_{\bar{M}_1 t^{\gamma-2/3}}^{\bar{M}_2 t^{\gamma-2/3}} dM\, MG(M) \int du \exp\left(\frac{-\beta M u^2}{2}\right)$$
$$\times S(Lt^{-2/3}-M-u+Y)\, S(Lt^{-2/3}-M+u-Y) \qquad (5.18)$$

Clearly, for $0<\gamma<2/3$, the behavior of (5.18) for large t is governed by that of $G(M)$ for small M. Since $G(M) \sim M^{-3/2}$ as $M \to \infty$ (see Appendix B), we obtain that (5.18) behaves asymptotically as

$$\frac{2L}{t^{2/3}} \int du\, S(u)\, S(-u) \int_{\bar{M}_1 t^{\gamma-2/3}}^{\bar{M}_2 t^{\gamma-2/3}} dM\, M^{-1/2}, \qquad t \to \infty \qquad (5.19)$$

implying

$$\mu^{(1)}(\bar{M}_1 t^\gamma, \bar{M}_2 t^\gamma; t \mid L) \geqslant C \frac{2L}{t^{1-\gamma/2}}, \qquad 0<\gamma<2/3 \qquad (5.20)$$

It is interesting to note that the estimate (5.20) interpolates between the result (5.14) (i.e., a nonvanishing probability of finding masses of order $t^{2/3}$) and the bound (5.17), which holds in fact for any aggregate composed of a fixed number of initial masses.

Summarizing the results just obtained for masses of smaller order than $t^{2/3}$, we can say that in an interval $(-L, L)$, the whole spectrum of masses $M \sim t^\gamma$ will be present with probabilities as large as $t^{-1+\gamma/2}$, $0 \leqslant \gamma \leqslant 2/3$. In particular, aggregates made of a finite number of initial masses will still be found at any time t with a probability not smaller than t^{-1}.

To conclude this section, we observe that (5.14), (5.17), and (5.20) have the following obvious implication for the number density $f(M, t)$ of particles of mass M:

$$f(M, t) = \frac{1}{2L} \int_{-L}^{L} dX \int_{-\infty}^{\infty} dP\, F(X, P, M, t) \qquad (5.21)$$

where F is defined in terms of the probability densities μ_k by (2.31). This definition implies clearly

$$\int_{M_1}^{M_2} dM\, f(M, t) \geqslant \frac{1}{2L} \mu^{(1)}(M_1, M_2; t \mid L) \qquad (5.22)$$

hence

$$\int_{\bar{M}_1 t^\gamma}^{\bar{M}_2 t^\gamma} dM\, f(M, t) \geq \frac{C}{t^{1-\gamma/2}} \tag{5.23}$$

In particular, the number density $f(t) = \int_0^\infty dM\, f(M, t)$ obeys the inequality

$$f(t) \geq \frac{C}{t^{2/3}} \tag{5.24}$$

in accordance with the fact discussed previously that the average distance between the particles can be at most of the order $t^{2/3}$.

6. CONCLUDING REMARKS

A full analysis of the (one-dimensional) ballistic aggregation process is far from having been completed by the present study. We comment on some open questions and possible developments.

Although the formulation of the problem given in Section 2 is quite general and allows for arbitrary initial states, we have focused our attention on a model of equal and equidistant initial masses with Maxwellian velocity distribution. For this model, our most significant results are the lower bounds to the asymptotic mass distribution for large time obtained in Section 5: we may conjecture that the behavior given by these lower bounds is also the exact one. As mentioned in the introduction, a proof of this conjecture requires better upper bounds than (4.1) and (4.2). The latter bounds are two crude because they do not take into account the effect of the barrier (3.12) in the neighborhood of $\tau = 0$ (i.e., for small masses).

The next task would be to obtain information on spatial and momentum correlations between different particles by an analysis of the higher-order probability distributions μ_k. Note that a more detailed knowledge of the particle density (5.21) requires a control of the whole series (2.31), and in particular suitable upper bounds on the μ_k. It would be interesting to confront the rigorous analysis with the conjecture suggested by the numerical simulations and the work of ref. 4 that the following limit exists:

$$\lim_{t \to \infty} t^{4/3} f(\bar{M} t^{2/3}, t) = \psi(\bar{M}) \tag{6.1}$$

and to determine the exact form of the scaling function $\psi(\bar{M})$.

In refs. 1 and 2 it has been advocated that the $t^{1/3}$ scaling for momentum results in the law of large numbers. We find indeed that masses of order $t^{2/3}$ propagate with momenta $t^{1/3}$ [at least in the sense of the estimate (5.13)]. However, there is no indication in (5.13) that the corresponding momentum distribution should be normal. This is not surprising in view of the kinematical constraints (elaborated in Section 2) necessary for the formation of an aggregate.

A peculiarity of our model is dealing with a single initial particle configuration. Still keeping the Maxwellian distribution of velocities, it would be natural to distribute the initial according to Poissonian statistics, i.e., to have the state of a free gas at the initial time. Because of the Maxwellian distribution, the mathematics of this model will still be reduced to the study of the Wiener integral of some functionals of the Brownian paths.

As a further generalization, we may allow for nonequilibrium momentum distributions. As explained in the beginning of Section 3, this involves now the determination of the asymptotic behavior of classes of functionals of more general random walks than Brownian motion. Such a generalization is necessary to establish the claim of universal long-time behavior supported by the simulations.

Another line of attack is to examine the implications of the present analysis for the structure of the particle correlation functions and the exact hierarchy of equations that governs their dynamics. Aggregation in an external force field and in higher dimension than one are also challenging problems. We believe that some of these questions are amenable to a mathematical control by the methods developed in this paper and deserve further study.

APPENDIX A

Let $f(\tau)$ be a twice differentiable function and consider the measure $K(M_1, P_1; M_2, P_2)$, (4.4), of the paths $P(\tau)$ that are above $f(\tau)$ for $M_1 \leqslant \tau \leqslant M_2$ [with $P(M_1) = P_1$, $P(M_2) = P_2$, $P_1 \geqslant f(M_1)$, $P_2 \geqslant f(M_2)$]. We introduce in (4.4) the paths $Z(\tau)$ translated by the time-dependent function $f(\tau)$,

$$Z(\tau) = P(\tau) - f(\tau) \tag{A.1}$$

Writing in formal terms that the Wiener measure is proportionnal to

$$\begin{aligned}
&\prod_{\tau = M_1}^{M_2} dP(\tau) \exp\left\{-\frac{\beta}{2}\int_{M_1}^{M_2} d\tau \left(\frac{dP(\tau)}{d\tau}\right)^2\right\} \\
&= \prod_{\tau = M_1}^{M_2} dZ(\tau) \exp\left\{-\frac{\beta}{2}\int_{M_1}^{M_2} d\tau \left[\left(\frac{dZ(\tau)}{d\tau}\right)^2 + 2\frac{dZ(\tau)}{d\tau}\frac{df(\tau)}{d\tau} + \left(\frac{df(\tau)}{d\tau}\right)^2\right]\right\}
\end{aligned} \tag{A.2}$$

and after an integration by parts, one obtains

$$\begin{aligned}
&K(M_1, P_1; M_2, P_2) \\
&\quad = \exp\left\{\beta\left(Z_1 f'(M_1) - Z_2 f'(M_2) - \tfrac{1}{2}\int_{M_1}^{M_2} d\tau\, (f'(\tau))^2\right)\right\} \\
&\quad \times E_w\left[\{Z(\tau) \geqslant 0,\ M_1 \leqslant \tau \leqslant M_2\}\right. \\
&\quad \times \exp\left(\beta \int_{M_1}^{M_2} d\tau\, Z(\tau) f''(\tau)\right) \left| Z(M_1) = Z_1; Z(M_2) = Z_2 \right] \qquad \text{(A.3)}
\end{aligned}$$

with $Z_1 = P_1 - f(M_1) \geqslant 0$, $Z_2 = P_2 - f(M_2) \geqslant 0$. The steps leading to (A.3) can be justified by performing them in the polygonal approximation of the Wiener integral.

According to the Feyman–Kac formula, the functional integral in (A.3) is given by the fundamental solution $G_D(M_1, Z_1; M_2, Z_2)$ of the differential equation on the half-line $Z_1 \geqslant 0$,[9]

$$\left(\frac{\partial}{\partial M_1} - \frac{1}{2\beta}\frac{\partial^2}{\partial Z_1^2} - \beta f''(M_1)\, Z_1\right) G_D(M_1, Z_1; M_2, Z_2) = 0 \qquad \text{(A.4)}$$

with $G_D(M_1, Z_1; M_2, Z_2)|_{M_1 = M_2} = \delta(Z_1 - Z_2)$ and Dirichlet boundary condition at $Z_1 = 0$,

$$G_D(M_1, 0; M_2, Z_2) = G_D(M_1, Z_1; M_2, 0) = 0, \qquad M_2 > M_1 \qquad \text{(A.5)}$$

In the case of the quadratic function (3.12) with $M_1 = 0$ and $M_2 = M$, we have $P_1 = Z_1$ and $P_2 = Z_2$ and (4.8) follows. Moreover, the corresponding differential operator H_D, (4.7), can be interpreted as the Schrödinger operator for a quantum mechanical particle of mass β constrained to a half-space and subjected to a uniform gravitationnal field of intensity $g = 2$ $(= \rho^{-1})$. The solution of this quantum mechanical problem is known in terms of Airy functions[10] and it could be used to derive the properties of (4.4). We will instead derive bounds in a direct way by the methods of Appendix B.

As a simple application of the formula (A.3), we consider the case of a linear barrier $f(\tau) = b\tau + d$. Then the solution of (A.4) is the free Dirichlet kernel

$$\begin{aligned}
&G_D^0(M_1, Z_1; M_2, Z_2) \\
&\quad \equiv G_D^0(M_2 - M_1, Z_1; Z_2) \\
&\quad = \left(\frac{\beta}{2\pi(M_2 - M_1)}\right)^{1/2} \left[\exp\left(-\frac{\beta(Z_1 - Z_2)^2}{2(M_2 - M_1)}\right) - \exp\left(-\frac{\beta(Z_1 + Z_2)^2}{2(M_2 - M_1)}\right)\right] \qquad \text{(A.6)}
\end{aligned}$$

and

$$K(M_1, P_1; M_2, P_2) = \exp\{\beta[b(Z_1 - Z_2) - \tfrac{1}{2}b^2(M_2 - M_1)]\} \times G_D^0(M_2 - M_1, Z_1; Z_2) \tag{A.7}$$

with $Z_1 = P_1 - bM_1 - d$, $Z_2 = P_2 - bM_2 - d$. In particular, the measure of the set of paths starting from P_1 at M_1 that are above $b\tau + d$ for all $\tau \geqslant M_1$ is

$$\begin{aligned}
&\lim_{M_2 \to \infty} \int_{bM_2 + d}^{\infty} dP_2\, K(M_1, P_1; M_2, P_2) \\
&= \exp(\beta b Z_1) \lim_{M \to \infty} \exp\left(\frac{-\beta b^2 M}{2}\right) \int_0^{\infty} dZ_2 \\
&\quad \times \exp(-\beta b Z_2)\, G_D^0(M, Z_1; Z_2) \\
&= \exp(\beta b Z_1) \lim_{M \to \infty} \exp\left(-\frac{\beta Z_1^2}{2M}\right) 2sh\left[\beta\left(\frac{Z_1}{\sqrt{M}} - b\right)\right] \\
&\quad \times \left(\frac{\beta}{2\pi}\right)^{1/2} \int_{b\sqrt{M}}^{\infty} du \exp\left(\frac{-\beta u^2}{2}\right) \\
&= \begin{cases} 1 - \exp(-2\beta\, |b|\, Z_1), & b \leqslant 0, \qquad Z_1 = P_1 - bM_1 - d \\ 0, & b \geqslant 0 \end{cases}
\end{aligned} \tag{A.8}$$

The second equality follows from the change of integration variable $Z_2 = \sqrt{M}\, u - bM$ after the introduction of (A.6).

APPENDIX B

We indicate the main steps to obtain the lower bound (4.13). Consider first the case $Y \geqslant Y_0$, $m \leqslant Y_0/4$, where Y_0 is some fixed positive number. The function $f_Y(\tau) = \tau(Y - \tau)$ is majorized by the polygonal line formed by its tangents at $\tau = m$ and $\tau = Y - m$ (see Fig. 1):

$$\begin{aligned}
l_1(\tau) &= (Y - 2m)\,\tau + m^2 \\
l_2(\tau) &= -(Y - 2m)\,\tau + (Y - m)^2
\end{aligned} \tag{B.1}$$

$$l_1(Y/2) = l_2(Y/2) = (Y^2/2) - m(Y - m), \qquad Y - 2m \geqslant Y_0/2 \tag{B.2}$$

The following inequality expresses the fact that the measure of the set of paths above $f_Y(\tau)$ is larger than that of the paths above the line formed by $l_1(\tau)$, $m \leqslant \tau \leqslant Y/2$, and $l_2(\tau)$, $\tau \geqslant Y/2$:

$$\lim_{M\to\infty}\int_{f(M)}^{\infty} dP_2\, K(m, P_1; M, P_2)$$

$$\geqslant \lim_{M\to\infty}\int_{l_2(M)}^{\infty} dP_2 \int_{l_1(Y/2)}^{\infty} dP\, K_1(m, P_1; Y/2, P)\, K_2(Y/2, P; M, P_2)$$

$$= \int_0^{\infty} dP\, K_1(m, P_1; Y/2, P + l_1(Y/2))\{1 - \exp[-2\beta P(Y-2m)]\}$$

$$= \exp[-\beta Y^2(Y-2m)/4]\exp[\beta P_1(Y-2m)] \tag{B.3}$$

$$\times \int_0^{\infty} dP \exp\{-\beta P(Y-2m)\}\{1-\exp[-2\beta P(Y-2m)]\}$$

$$\times G_D^0((Y-2m)/2, P_1 - m(Y-m); P) \tag{B.4}$$

In (B.3), $K_1(m, P_1; Y/2, P)$ [resp. $K_2(Y/2, P; M, P_2)$] is the conditional measure of the set of paths that are above $l_1(\tau)$, $m \leqslant \tau \leqslant Y/2$ [resp. above $l_2(\tau)$, $\tau \geqslant Y/2$]. The first equality results from (A.8), and we have written K_1 with the help of the formula (A.7). Since G_D^0 is given by (A.6), the lower bound (B.4) is an explicit function of P_1, Y, and m. In order to use it in (4.12) we must set $P_1 = \sqrt{m}\,u$ in (B.4) and estimate it as $m \to 0$. This

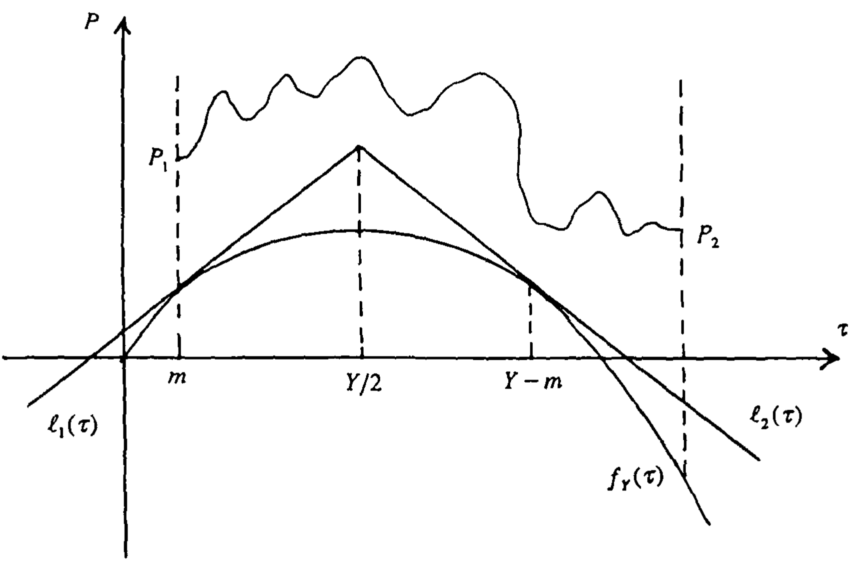

Fig. 1. Brownian motion above a parabolic barrier: the case $Y > 0$.

involves expanding $G_D^0(M, P_0; P)$ in the neighborhood of $P_0 = 0$. A limited Taylor expansion of (A.6) leads to

$$G_D^0(M, P_0; P) = P_0 \frac{\partial}{\partial P_0} G(M, P_0; P)\bigg|_{P_0=0} + P_0^2 O(P+1) \qquad \text{(B.5)}$$

provided that M remains bounded away from the origin. This gives, with $P_0 = \sqrt{m}\, u - m(A-m)$ and $M = Y/2 - m \geqslant Y_0/4 > 0$,

$$\begin{aligned} &G_D^0((Y-2m)/2, P_1 - m(Y-m); P) \\ &\quad = \sqrt{m}\, u(\beta^3/\pi Y^3)^{1/2}\, 2P \exp(-\beta P^2/Y) + mR(u, Y, P, m) \qquad \text{(B.6)} \end{aligned}$$

where the rest R is polynomially bounded in all its variables. We now insert (B.6) into (B.4), and then this whole expression into (4.12), and expand in the neighborhood of $m = 0$. The dominant term is obviously of order $\sqrt{m}$, and the rest of the expansion is under control because of the following reasons:

(i) The P integral is convergent because of the factor $\exp[-\beta P(Y-2m)]$.

(ii) The u integral is convergent because of the Gaussian $\exp(-\beta u^2/2)$.

(iii) An possible growth with Y is killed by the prefactor $\exp(-Y^3\beta/4)$ as $Y \to \infty$. For instance, one uses ($k \geqslant 0$)

$$\begin{aligned} &\exp(-Y^3\beta/4) \int_0^\infty du\, u^k \exp[-\beta(u^2/2 - \sqrt{m}\, uY)] \\ &\quad = \exp(-Y^3\beta/4) \int_0^\infty du\, u^k \exp(-\beta u^2/2) + \sqrt{m}\, O(\exp(-CY^3)) \qquad \text{(B.7)} \end{aligned}$$

The result of the estimate is, under the condition (B.2),

$$J_m \geqslant (\beta m/2\pi)^{1/2} S(Y) + mO(\exp(-CY^3)) \qquad \text{(B.8)}$$

with

$$\begin{aligned} S(Y) = {} &\exp(-Y^3\beta/4)\, 2(\beta^3/\pi Y^3)^{1/2} \int_0^\infty dP \exp(-\beta PY) \\ &\times [1 - \exp(-2\beta PY)]\, P \exp(-\beta P^2/Y) \qquad \text{(B.9)} \end{aligned}$$

We consider now the case where $0 \leqslant Y \leqslant Y_0$ and $m \leqslant Y/2$. We have to proceed differently because of the singularity of the Dirichlet kernel

$G(M, P_0; P_1)$ at $M = Y/2 - m = 0$. For this, one introduces in (B.4) the majoration $1 - \exp[-2\beta P(Y - 2m)] \leqslant 2\beta P(Y - 2m)$ and performs again a limited Taylor expansion. The additional factor $Y - 2m$ precisely compensates for the singularity of the Dirichlet kernel. One obtains again that the remainder is finite, and that the estimate holds also in this case. Notice in (B.9) that $S(Y) = O(Y)$, $Y \to 0$.

It remains to examine the case $Y \leqslant 0$. We majorize $f_Y(\tau)$ by its tangent $l_1(\tau)$ at $\tau = m$ (Fig. 2)

$$l_1(\tau) = -(|Y| + 2m)\,\tau + m^2 \tag{B.10}$$

One has the obvious inequality

$$\begin{aligned}
&\lim_{M \to \infty} \int_{f_Y(\tau)}^{\infty} dP_2\, K(m, P_1; M, P_2) \\
&\geqslant \lim_{M \to \infty} \int_{l_1(M)}^{\infty} dP_2\, K(m, P_1; M, P_2) \\
&= 1 - \exp\{-2\beta(|Y| + 2m)[P_1 + m(|Y| + m)]\} \\
&\geqslant 1 - \exp[-2\beta(|Y| + 2m)\, P_1]
\end{aligned} \tag{B.11}$$

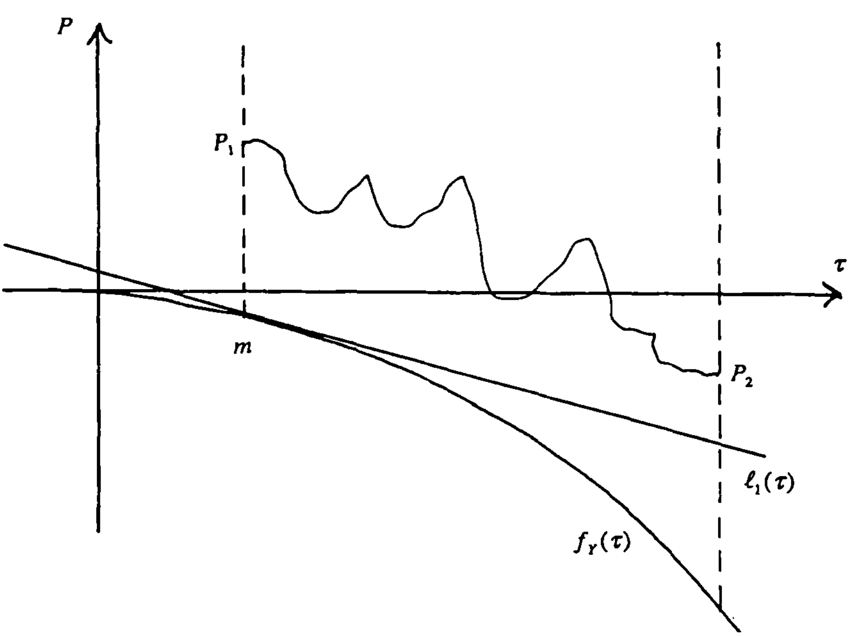

Fig. 2. Brownian motion above a parabolic barrier: the case $Y < 0$.

where the equality results from (A.8). When (B.11) is inserted into (4.12) with $p_1 = \sqrt{m}\, u$, one obtains

$$J_m(Y) \geq (\beta m/2\pi)^{1/2}\, S(Y) + mO(Y^2+1) \qquad \text{(B.12)}$$

with $S(Y) = 2\beta\,|Y|$.

In order to establish (4.11), we proceed as in the situation of Fig. 1 with $f_Y(\tau)$ replaced by $f_M(\tau)$ $(0 \leq \tau \leq M)$. One obtains the inequality

$$\begin{aligned} &K(m, P_1; M-m, P_2) \\ &\quad \leq \int_{l_1(M/2)}^{\infty} dP\, K_1(m, P_1; M/2, P)\, K_2(M/2, P; M-m, P_2) \qquad \text{(B.13)} \end{aligned}$$

The quantities K_1 and K_2 have the same meaning as in (B.3). Both of them can be explicitly expressed in terms of the Dirichlet kernel (A.6) with the help of (A.7). Once this is done, one sets $p_1 = \sqrt{m}\, u$ and $p_1 = \sqrt{m}\, v$ [see (4.5)] and performs a limited Taylor expansion. For $M \geq M_0 > 0$ and m small enough, one can estimate the two Dirichlet kernels occurring in (B.13) by formulas corresponding to (B.5) and (B.6). Clearly, the dominant term is of the order muv with a remainder $O(m^{3/2})$. This justifies (4.9) and one can work out an explicit positive lower bound for $G(M)$ of the form

$$G(M) \geq C_1 M^{-3/2} \exp(-C_2 M^3) \qquad \text{(B.14)}$$

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